3 Sobolev Spaces

Exercise 3.1. Let $\delta > 0$ and consider the collection $\{Q_i : i \in \mathbb{N}\}$ where Q_i is an n-dimensional cube of side-length $\delta > 0$ such that the center of Q_i is a point $x = (x_1, \dots, x_n)$ such that $x_i/\delta \in \mathbb{Z}$ for every $i = 1 \dots d$. Clearly $\mathbb{R}^d = \bigcup_i Q_i$. Given $u \in L^1_{loc}(\mathbb{R})$ and $\delta > 0$, we set

$$u_\delta(x) = \frac{1}{\delta^d} \int_{Q_i} u(y)dy \qquad \text{for all } x \in Q_i.$$

Show that whenever $u \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$, then for every $\delta > 0$ the function $u_\delta \in L^p(\mathbb{R}^d)$, moreover

$$\|u_\delta\|_{L^p(\mathbb{R}^d)} \leq \|u\|_{L^p(\mathbb{R}^d)}$$

and $u_\delta \to u$ in $L^p(\mathbb{R}^d)$ as $\delta \to 0$.

Exercise 3.2. Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^d$ be open, with finite measure and such that there exists an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$. For every $u \in W^{1,p}(\Omega)$ let u_δ be the function defined in Exercise 1.

— Show that if $\mathfrak{F} \subset L^p(\Omega)$ is bounded, then the following equivalence holds :
  $\mathfrak{F}$ is relatively compact if and only if for every $\varepsilon > 0$ there exists some $\overline{\delta} > 0$ such that for all $\delta < \overline{\delta}$ and all $u \in \mathfrak{F}$ one has $\|u - u_\delta\|_{L^p(\Omega)} < \varepsilon$.

— Prove that for every $\delta > 0$ one has

$$\|u - u_\delta\|_{L^1(\Omega)} \leq 2^d \sqrt{d}\delta \|Du\|_{L^1(\Omega,\mathbb{R}^d)}.$$

Exercise 3.3 (Poincaré Inequality)**.** . Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be open and bounded at least in one direction, i.e. $\Omega \subset (0, \ell) \times \mathbb{R}^{n-1}$ for some $\ell > 0$ up to a rotation and a translation. Show that there exists a constant $C > 0$ such that

$$\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega,\mathbb{R}^n)} \qquad \text{for every } u \in W^{1,p}_0(\Omega).$$

Exercise 3.4 (Another variant of Poincaré Inequality)**.** . Let $1 \leq p < \infty$ and $\Omega = B_1(0) \subset \mathbb{R}^d$. Given $\alpha > 0$, show that there exists a constant $C > 0$ depending only on $d, p, \alpha > 0$ such that for every $u \in W^{1,p}(\Omega)$ such that $|\{x : u(x) = 0\}| \geq \alpha$ it holds

$$\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}.$$

Exercise 3.5. Let $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^d$ be an open set. Denote by

$$W^{1,p}_{\mathrm{div}}(\Omega) = \{u = (u_1, \dots, u_n) \in (L^p(\Omega))^d : \mathrm{div}\, u \in L^p(\Omega)\}.$$

Show that $W^{1,p}_{\mathrm{div}}(\Omega)$ is a Banach space if endowed with the norm

$$\|u\| = \sum_{i=1}^{s} \|u_i\|_{L^p(\Omega)} + \|\mathrm{div}\, u\|_{L^p(\Omega)}.$$

Exercise 3.6. For $0 < \theta < 1$ and $1 \leq p < \infty$ we call $W^{\theta,p}(\mathbb{R}^d)$ the space of all functions $f \in L^p(\mathbb{R}^s)$ such that

$$|f|_{W^{\theta,p}} = \left(\int_{\mathbb{R}^s \times \mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x - y|^{d+\theta p}} \, dx \, dy\right)^{1/p} < +\infty.$$

Prove that $W^{\theta,p}(\mathbb{R}^s)$ is a Banach space with respect to the norm $\|\cdot\|_{W^{\theta,p}} = \|\cdot\|_{L^p} + |\cdot|_{W^{\theta,p}}$.